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OPERATIONAL CALCULUS ON WAVELET TRANSFORM AS AN EXTENSION OF FRACTIONAL FOURIER TRANSFORM

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Abstract: Integral transforms have been in wide use to solve various differential equations or problems in pure and applied mathematics. Wavelet Transform and Fractional Fourier transform has many applications in signal and image processing.

In this paper describe the various properties like Linearity, Translation, differentiation of Wavelet Transform as an extension of Fractional Fourier transform they will be useful for solving differential and integral equation.

Keywords: Fractional Fourier Transform, Wavelet Transform, Extended Wavelet Transform.
Mathematics Subject Classification: 44A05

1. Introduction:

Mathematics is everywhere in every technology, subject, experiment, etc. but we necessary to find logic behind it[5]. Integral Transform was successfully used for almost 200 years for solving many problems in mathematics and physics[6]. There are many integral transforms have been used for solving differential equations. The fractional Fourier analysis is used for investigations of fractal structures; which in turn are used to analyze different physical phenomena[2]. The ordinary Fourier transform and related techniques are of great importance in many areas of science and engineering[9]. The Fourier transform is best mathematical tool used in differential equations, physical optics, signal and image processing and so on[1,4].

The concept of wavelet started to appeared in the literature only in the 19^{th} century 8^{th} decade that used by Morlet(1982)[3,8]. A French geophysical engineering first introduced the idea of wavelet transform as the mat hematical tool for signal and image processing[5]. The wavelet transform decomposes a signal into the representation that shows signal details and tends as a function of time. The kernel of fractional Fourier transform and wavelet transform are nearly related and Sharma and bhosale introduce the Wavelet transform as an extension of fractional Fourier transform[7]. so we are going to discuss Properties of Wavelet transform as an extension of fractional Fourier transform.

2. Priliminaries:

Wavelet Transform as an extension of Fractional Fourier Transform[5]:

The Wavelet transform as the extension of fractional Fourier transform of $f(x) \in E(R^n)$ is denoted by $W(f(x))(\xi)$ and defined by,

$$W(f(x))(a,b) = W(f(x))(\xi) = \int_{-\infty}^{\infty} BC_{1\alpha} f(x) e^{iC_{2\alpha}[(x^2+\xi^2)cos\alpha - 2x\xi]}\, dx$$

Where, $b = \xi sec\alpha$, $a = \tan^{\frac{1}{2}}\alpha$, $B = \frac{e^{\frac{i}{2}} b^2 \sin^2\alpha}{C(\alpha)|\alpha|^n}$, $C(\alpha) = \frac{e^{\frac{i\alpha}{2}}}{(2\pi isin\alpha)^{\frac{1}{2}}}$, $C_{2\alpha} = \frac{1}{2sin}$,

$C_{1\alpha} = (2\pi isin\alpha)^{-\frac{1}{2}} \exp\left(\frac{i\alpha}{2}\right)$, $0 \le \alpha < \frac{\pi}{2}$.

Testing Function Space $E(R^n)$:

An infinitely differentiable complex valued function *f* on R^n belongs to $E(R^n)$ if for each compact set $X \subset S_\beta$ where

$S_\beta = \{y \in R^n : |y| \le \beta, \beta > 0\}$

3. Results:

Linearity Property of Extended Wavelet Transform:

$W(af(x) + bg(x))(\xi) = aW(f(x) + bW(g(x))$

Proof: We Know that,

$$W(f(x))(\xi) = \int_{-\infty}^{\infty} BC_{1\alpha} f(x) e^{iC_{2\alpha}[(x^2+\xi^2)cos\alpha - 2x\xi]}\, dx$$

$W(af(x) + bg(x))(\xi) = \int_{-\infty}^{\infty} BC_{1\alpha} e^{iC_{2\alpha}[(x^2+\xi^2)cos\alpha - 2x\xi]} \left(af(x) + bg(x)\right) dx$

$= a\int_{-\infty}^{\infty} BC_{1\alpha} e^{iC_{2\alpha}[(x^2+\xi^2)cos\alpha - 2x\xi]} f(x)dx +$

$b\int_{-\infty}^{\infty} BC_{1\alpha} e^{iC_{2\alpha}[(x^2+\xi^2)cos\alpha - 2x\xi]}\, g(x)dx$

$= aW(f(x) + bW(g(x))$

Extended Wavelet Transform of Translation:

$$W(f(x - x_0))(\xi) = e^{iC_{2\alpha}[x_0^2 cos\alpha \quad _0\xi]} W\left(e^{(2ic_{2\alpha}x_0 cos\alpha)x} f(x)\right)(\xi)$$

Proof: $W(f(x - x_0))(\xi) = \int_{-\infty}^{\infty} BC_{1\alpha} e^{iC_{2\alpha}[(x^2+\xi^2)cos\alpha - 2x\xi]} f(x - x_0)\, dx$

Put $x - x_0 = t$, then $x = x_0 + t$, $dx = dt$

$W(f(x - x_0))(\xi) = \int_{-\infty}^{\infty} BC_{1\alpha} e^{iC_{2\alpha}[((x_0+t)^2+\xi^2)cos\alpha - 2(x_0+t)\xi]} f(t)\, dt$

$= e^{iC_{2\alpha}[x_0^2 cos\alpha - 2x_0\xi]} \int_{-\infty}^{\infty} BC_{1\alpha} e^{iC_{2\alpha}[(t^2+\xi^2)cos\alpha - 2t\xi]} e^{iC_{2\alpha}[x_0 cos\alpha]} f(t)\, dt$

$= e^{iC_{2\alpha}[x_0^2 cos\alpha - 2x_0\xi]} \int_{-\infty}^{\infty} BC_{1\alpha} e^{iC_{2\alpha}[(x^2+\xi^2)cos\alpha - 2x\xi]} e^{iC_{2\alpha}[x_0 cos\alpha]} f(x)\, dx$

$= e^{iC_{2\alpha}[x_0^2 cos\alpha - 2x_0\xi]} W\left(e^{(2ic_{2\alpha}x_0 cos\alpha)x} f(x)\right)(\xi)$

Translation of Extended Wavelet Transform:

$$W\big(f(x)\big)(\xi-\xi_0)=e^{iC_{2\alpha}[\xi_0^2-2\xi_0\xi]cos\alpha}W\left(e^{(2iC_{2\alpha}\xi_0)x}f(x)\right)(\xi)$$

Proof:

$$W\big(f(x)\big)(\xi-\xi_0)=\int_{-\infty}^{\infty}BC_{1\alpha}e^{iC_{2\alpha}[(x^2+(\xi-\xi_0)^2)cos\alpha-2x(\xi-\xi_0)]}f(x)\,dx$$

$$=e^{iC_{2\alpha}[(\xi_0^2-2\xi_0\xi)cos\alpha]}\int_{-\infty}^{\infty}BC_{1\alpha}e^{iC_{2\alpha}[(x^2+\xi^2)cos\alpha-2x\xi]}e^{2iC_{2\alpha}\xi_0x}f(x)\,dx$$

$$=e^{iC_{2\alpha}[\xi_0^2-2\xi_0\xi]cos\alpha}W\left(e^{(2iC_{2\alpha}\xi_0)x}f(x)\right)(\xi)$$

Differentiation of Extended Wavelet Transform:

$$D^nW\big(f(x)\big)(\xi)=\frac{d^n}{d\xi^n}W\big(f(x)\big)(\xi)=W\left(\sum_{h=0}^{\left[\frac{n}{2}\right]}C_hC_{\alpha,h}(\xi cos\alpha-x)^{n-2h}f(x)\right)(\xi)$$

Where, $C_h=\frac{n!}{(n-2h)!h!}(i)^{n-h}(2)^{n-2h}, C_{\alpha,h}=(C_{2\alpha})^{n-h}\cos^{h}\alpha$

Proof:

For $n=1$

$$\frac{d}{d\xi}W\big(f(x)\big)(\xi)=\int_{-\infty}^{\infty}\frac{\partial}{\partial\xi}BC_{1\alpha}f(x)e^{iC_{2\alpha}[(x^2+\xi^2)cos\alpha-2x\xi]}\,dx$$

$$=2iC_{2\alpha}\xi cos\alpha\int_{-\infty}^{\infty}\frac{\partial}{\partial\xi}BC_{1\alpha}f(x)e^{iC_{2\alpha}[(x^2+\xi^2)cos\alpha-2x\xi]}\,dx$$

$$-2iC_{2\alpha}\int_{-\infty}^{\infty}\frac{\partial}{\partial\xi}BC_{1\alpha}xf(x)e^{iC_{2\alpha}[(x^2+\xi^2)cos\alpha-2x\xi]}\,dx$$

$$=2iC_{2\alpha}(\xi cos\alpha-x)W\big(f(x)\big)(\xi)=W\left(\sum_{h=0}^{\left[\frac{1}{2}\right]}C_hC_{\alpha,h}(\xi cos\alpha-x)^{1-2h}f(x)\right)(\xi)$$

For $n=2$

$$\frac{d^2}{d\xi^2}W\big(f(x)\big)(\xi)=\frac{d}{d\xi}\left[2iC_{2\alpha}\xi cos\alpha W\big(f(x)\big)(\xi)-2iC_{2\alpha}W\big(xf(x)\big)(\xi)\right]$$

$$=2iC_{2\alpha}\xi cos\alpha\frac{d}{d\xi}W\big(f(x)\big)(\xi)+2iC_{2\alpha}cos\alpha W\big(f(x)\big)(\xi)-2iC_{2\alpha}\frac{d}{d\xi}W\big(xf(x)\big)(\xi)$$

$$=2iC_{2\alpha}cos\alpha W\big(f(x)\big)(\xi)+4i^2C_{2\alpha}^2\xi^2\cos^2\alpha\, W\big(f(x)\big)(\xi)-4i^2C_{2\alpha}^2\xi cos\alpha W\big(xf(x)\big)(\xi)$$

$$-4i^2C_{2\alpha}^2\xi cos\alpha W\big(xf(x)\big)(\xi)+4i^2C_{2\alpha}^2W\big(x^2f(x)\big)(\xi)$$

$$=W\left(\sum_{h=0}^{\left[\frac{2}{2}\right]}C_hC_{\alpha,h}(\xi cos\alpha-x)^{2-2h}f(x)\right)(\xi)$$

Continuing in this way we get,

$$D^n W\big(f(x)\big)(\xi) = \frac{d^n}{d\xi^n} W\big(f(x)\big)(\xi) = W\left(\sum_{h=0}^{\left[\frac{n}{2}\right]} C_h C_{\alpha,h} (\xi cos\alpha - x)^{n-2h} f(x)\right)(\xi)$$

Where, $C_h = \frac{n!}{(n-2h)!h!} (i)^{n-h} (2)^{n-2h}, C_{\alpha,h} = (C_{2\alpha})^{n-h} \cos^{\mathrm{h}} \alpha$

[n] denotes greatest integer function

Similarly, we show Extended Wavelet Transform of a Derivative

Extended Wavelet Transform of a Derivative:

$$W\big(f^n(x)\big)(\xi) = W\left((-1)^n \sum_{h=0}^{\left[\frac{n}{2}\right]} C_h C_{\alpha,h} (x cos\alpha - \xi)^{n-2h} f(x)\right)(\xi)$$

Where, $C_h = \frac{n!}{(n-2h)!h!} (i)^{n-h} (2)^{n-2h}, C_{\alpha,h} = (C_{2\alpha})^{n-h} \cos^{\mathrm{h}} \alpha$

4. Conclusion:

This paper presents some properties of Wavelet transform as an extension of fractional Fourier transform and this property are useful to solve ordinary differential equations and partial differential equations like heat equation, schrodinger's equation etc.

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